Math 656 • FINAL EXAM • May 13, 2014

In all problems below, use the branch $-\pi \le \arg z < \pi$ for multivalued functions, unless specified otherwise

1) (8pts) Find **all values** of $\tanh^{-1}(i)$.

2) (12pts) Categorize **all singularities** of the following functions. Examine also a possible singularity at $z=\infty$ (hint: substitute $\zeta = 1/z$). Make sure to explain briefly.

(a) $f(z) = \dfrac{1}{z^{1/4}\sin z}$ (b) $f(z) = \dfrac{\exp(z)}{\exp(1/z)}$ (c) $f(z) = \dfrac{\sin(\pi z)}{\sin^2(\pi/z)}$

3) (12pts) Find the first **two** dominant terms in the series expansion of $f(z) = \dfrac{\cos(\log_p(z)) - 1}{\sin \pi z}$ around $z = 1$. Hint: a shift $z = 1 + \zeta$ may help. What would be the radius of convergence of the full series around $z=1$?

4) (16pts) Calculate the following integrals, picking the most efficient method for each. Contours are circles of given radius:

(a) $\displaystyle\oint_{|z|=1} \frac{dz}{(\bar{z})^{1/4}}$ (b) $\displaystyle\oint_{|z|=2} \frac{\exp(1/z)}{1-z^2}\,dz$

5) (16pts) Calculate the following two integrals. Carefully explain each step, and make sure to obtain a **real** answer.

(a) $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$ (b) $\displaystyle\int_0^\infty \frac{x^3\,dx}{x^6 + a^6}$ (a is a real constant)

6) (12pts) Use Rouche's Theorem to find the number of zeros of $f(z) = 4z^4 + 13z^2 + 3$ belonging to the following domains: (a) $|z| < 1$; (b) $|z| < 2$; (c) $1 < |z| < 2$

Do two of the last four problems:

7) (12pts) Use the Argument Principle to find the number of roots of $f(z) = 2i - z + z^2 + z^3$ lying in the first quadrant. To do this, sketch the mapping of the relevant quarter-circle (it's quite straightforward).

8) (12pts) Suppose $f(z)$ is an entire function, satisfying inequality $|f(z)| < a + |z|^k$ everywhere in the complex plane (here $a > 0$ is a real constant). Prove that $f(z)$ is a polynomial. Hint: recall the proof of the Liouville's Theorem using the extended version of Cauchy Integral Formula.

9) (12pts) Indicate domains of convergence of each series:

a) $\displaystyle\sum_{k=0}^{\infty} \frac{\exp(2zk)}{k!}$ b) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{\exp(-zk)}{k}$

10) (12pts) Consider the map $w = z + \dfrac{1}{z}$. Describe the images of the following sets under this map: **(a)** unit circle $|z|=1$, **(b)** circle of radius 2, $|z|=2$. **(c)** exterior of the unit disk, $|z|>1$. Hint: examine Cartesian components of the image, $w = u + i\,v$